arXiv:math/0605259v1 [math.GN] 10 May 2006

AXIOMATIZATION OF TOPOLOGICAL SPACE IN TERMS OF THE OPERATION OF BOUNDARY

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Abstract. We present the set of axioms for topological space with the operation of boundary as primitive notion.

Let X denote space (without any ascribed structure) and $\mathcal{P}(X)$ family of its subsets. We say that $\overline{(\cdot)} : \mathcal{P}(X) \to \mathcal{P}(X)$ is *closure operation* if for any sets $A, B \subset X$

(δ-1) $\overline{\emptyset} = \emptyset$,
(δ-2) $\overline{\overline{A}} \subset \overline{A}$,
(δ-3) $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$,
(δ-4) $A \subset B \Rightarrow \overline{A} \subset \overline{B}$,
(δ-5) $A \subset \overline{A}$.

We say that $\partial : \mathcal{P}(X) \to \mathcal{P}(X)$ is *operation of boundary* if for any sets $A, B \subset X$

(β-1) $\partial\emptyset = \emptyset$,
(β-2) $\partial\partial A \subset \partial A$,
(β-3) $\partial(A \cup B) \subset \partial A \cup \partial B$,
(β-4) $A \subset B \Rightarrow \partial A \subset B \cup \partial B$,
(β-5) $\partial A = \partial(X \setminus A)$.

Axiom (β-5) can be still weakened to

(β-5') $\partial A \subset \partial(X \setminus A)$.

Below we give natural correspondence between these notions. Define $\Phi : \mathcal{P}(X)^{\mathcal{P}(X)} \to \mathcal{P}(X)^{\mathcal{P}(X)}$, $\forall_{\mathrm{op} \in \mathcal{P}(X)^{\mathcal{P}(X)}} \forall_{A \in \mathcal{P}(X)} \ [\Phi(\mathrm{op})](A) \doteq \mathrm{op}(A) \cap \mathrm{op}(X \setminus A)$, and $\Psi : \mathcal{P}(X)^{\mathcal{P}(X)} \to \mathcal{P}(X)^{\mathcal{P}(X)}$, $\forall_{\mathrm{op} \in \mathcal{P}(X)^{\mathcal{P}(X)}} \forall_{A \in \mathcal{P}(X)} \ [\Psi(\mathrm{op})](A) \doteq A \cup \mathrm{op}(A)$.

Proposition 1 (boundary via closure). *If $\overline{(\cdot)}$ is closure operation, then $\Phi(\overline{(\cdot)})$ is operation of boundary.*

Proof. All calculations are standard so we show for example that $\partial \doteq \Phi(\overline{(\cdot)})$ satisfies (β-4). If $A \subset B$, then $\overline{A} \subset \overline{B}$ in view of (δ-4). Hence $\partial A = \overline{A} \cap \overline{X \setminus A} \subset \overline{A} \subset \overline{B}$. Further

$$\overline{B} \subset \overline{B} \cup (X \setminus \overline{X \setminus B}) = (\overline{B} \cap \overline{X \setminus B}) \cup (X \setminus \overline{X \setminus B}) \overset{(*)}{\subset} (\overline{B} \cap \overline{X \setminus B}) \cup B = B \cup \partial B,$$

Date: May 9, 2006.

2000 *Mathematics Subject Classification.* 54A05.
Key words and phrases. Axioms, topological space, operation of boundary, closure operation.

where inclusion $(*)$ is due to (δ-5). $\boxtimes$

Proposition 2 (closure via boundary)**.** *If ∂ is operation of boundary, then $\Psi(\partial)$ is closure operation.*

Proof. Since most calculations are straightforward we only demonstrate that $\overline{(\cdot)} \doteq \Psi(\partial)$ fulfills (δ-2) and (δ-4).

ad (δ-2): $\overline{\overline{A}} = \overline{A \cup \partial A} = (A \cup \partial A) \cup \partial(A \cup \partial A) \overset{(*)}{\subset} A \cup \partial A \cup \partial\partial A \overset{(**)}{\subset} A \cup \partial A = \overline{A}$, where $(*)$ uses (β-3) and $(**)$ uses (β-2).

ad (δ-4): if $A \subset B$, then $\partial A \subset B \cup \partial B$ by (β-4). Hence $\overline{A} = A \cup \partial A \subset B \cup \partial B = \overline{B}$. $\boxtimes$

Denote by $\mathcal{C}, \mathcal{B} \subset \mathcal{P}(X)^{\mathcal{P}(X)}$ the family of closure and respectively boundary operations.

Proposition 3 (equivalence of definitions)**.** *The correspondences $\Phi : \mathcal{C} \to \mathcal{B}$ and $\Psi : \mathcal{B} \to \mathcal{C}$ are mutually inverse. In particular Φ and Ψ are bijections.*

Proof. Let $\overline{(\cdot)} \in \mathcal{C}$, $A \in \mathcal{P}(X)$ and $\partial \doteq \Phi(\overline{(\cdot)})$. Then $\Psi(\Phi(\overline{(\cdot)}))\,(A) = A \cup \partial A = A \cup (\overline{A} \cap \overline{X \setminus A}) \overset{(*)}{\subset} \overline{A}$, where $(*)$ uses (δ-5). To get the reverse inclusion in $(*)$ axiom (δ-5) is used again: $\overline{A} \setminus A \subset X \setminus A \subset \overline{X \setminus A}$, so $\overline{A} = A \cup (\overline{A} \setminus A) \subset A \cup \overline{X \setminus A}$.

Now let $\partial \in \mathcal{B}$, $A \in \mathcal{P}(X)$ and $\overline{(\cdot)} \doteq \Psi(\partial)$. Then $\Phi(\Psi(\partial))\,(A) = \overline{A} \cap \overline{X \setminus A} = (A \cup \partial A) \cap ((X \setminus A) \cup \partial(X \setminus A)) \overset{(*)}{=} (A \cup \partial A) \cap ((X \setminus A) \cup \partial A) = (A \cap (X \setminus A)) \cup \partial A = \partial A$, where $(*)$ uses (β-5). $\boxtimes$

Observe that axiom (β-5) used to prove Proposition 3 is not exploited in the proof of Proposition 2. Nevertheless this axiom is indispensable as claimed by

Proposition 4 (logical independence)**.** *The system of axioms (β-1) – (β-5) is logically independent.*

We split the verification of the above proposition in the series of examples.

Example 1. Put $\partial_1(A) \doteq X$ for any $A \subset X$. Then ∂_1 fulfills all axioms of boundary except (β-1). $\Diamond$

Example 2. Let $X \doteq \mathbb{N}$ and

$$\partial_2(A) \doteq \left\{ x \in \mathbb{N} \,:\, \inf_{a \in A} |x - a| = 1 \vee \inf_{b \in \mathbb{N} \setminus A} |x - b| = 1 \right\}$$

for any $A \subset X$. Then ∂_2 fulfills all axioms of boundary except (β-2). $\Diamond$

Example 3. Let $X \doteq \{1, 2, 3\}$ and

$$\partial_3(A) \doteq \begin{cases} \emptyset, & \text{if } A = \emptyset \text{ or } X, \\ A, & \text{if } A = \{1\} \text{ or } \{2\} \text{ or } \{3\}, \\ X \setminus A, & \text{if } A = \{1,2\} \text{ or } \{2,3\} \text{ or } \{1,2\}, \end{cases}$$

for any $A \subset X$. Then ∂_3 fulfills all axioms of boundary except (β-3). $\lozenge$

Example 4. Let $X \doteq \{1, 2, 3\}$ and

$$\partial_4(A) \doteq \begin{cases} \emptyset, & \text{if } A = \emptyset \text{ or } X, \\ \{2\}, & \text{if } A = \{1\} \text{ or } \{2,3\}, \\ \{1\}, & \text{if } A = \{2\} \text{ or } \{1,3\}, \\ \{1,2\}, & \text{if } A = \{3\} \text{ or } \{1,2\}. \end{cases}$$

for any $A \subset X$. Then ∂_4 fulfills all axioms of boundary except (β-4). $\lozenge$

Example 5. Fix $x_0 \in X$ and put $\partial_5(A) \doteq A \cup \{x_0\}$ for every nonempty $A \subset X$ and $\partial_5(\emptyset) = \emptyset$. Then ∂_5 fulfills all axioms of boundary except (β-5). $\lozenge$

In the context of the last example we remark that any closure operation satisfies (β-1) – (β-4) but never (β-5).

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